JOURNAL OF COMPUTATIONAL PHYSICS 49, 465–477 (1983)

Numerical Experiments Involving Galerkin and Collocation Methods for Linear Integral Equations of the First Kind*

RICHARD C. ALLEN, JR.

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131

AND

W. ROBERT BOLAND AND G. MILTON WING

Computer Research and Applications, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

Received May 19, 1982

Recently efforts have been made to quantify the difficulties inherent in numerically solving linear Fredholm integral equations of the first kind (*J. Integral Equations*, to appear.). In particular, the classical quadrature approach, collocation methods, and Galerkin schemes that make use of various orthonormal basis functions have been shown to lead to matrices with high condition numbers. In fact, it has been possible to obtain explicit lower bounds on these condition numbers as a function of the smoothness of the kernel, essentially independent of the choice of orthonormal basis. These bounds all approach infinity as the number of basis functions increases. In this article we present a numerical study of condition numbers arising from collocation and Galerkin methods with step-function and Legendre polynomial bases. The condition number for each kernel and basis set studied is exhibited as a function of the number of basis functions used. The effect that these ill-conditioned matrices have on the accuracy of solutions is demonstrated computationally. The information obtained gives an indication of the efficacy—and the dangers—of the collocation and Galerkin schemes in practical situations.

I. INTRODUCTION

Linear Fredholm integral equations of the first kind arise quite often in applications. Many experimental techniques give rise naturally to such equations, and the subsequent attempt to interpret the experiment leads to a need to solve the equation. We shall examine some of the inherent difficulties involved.

* This work was performed under the auspices of the U.S. Department of Energy.

0021-9991/83 $3.00

Such equations have the form

$$g(x) = \int_a^b K(x, y) f(y)\, dy, \tag{1.1}$$

where the kernel $K(x, y)$ and the function $g(x)$ are assumed known and the problem is to find $f(y)$. As an example of a physical situation in which an equation of this type arises, consider the following experiment: Suppose a signal in the form of a beam of particles is passed through a slab of absorbing but nonscattering material. Assume that the known absorption cross section, $\sigma(E)$, of the slab depends only on the particle energy. The output of the experiment is a single number provided by a detector that measures the total number of particles impinging on it. The problem is to determine the energy spectrum of the incident particle beam. Let x denote the thickness of the slab. If $f(E)$ is the energy spectrum and $g(x)$ the detector measurement for thickness x, the problem can be formulated mathematically as

$$g(x) = \int_{E_{\min}}^{E_{\max}} e^{-\sigma(E)x} f(E)\, dE. \tag{1.2}$$

It is physically clear that a single experiment cannot determine the energy spectrum $f(E)$. If, however, the experiment is repeated for various thickness x, then Eq. (1.2) is simply Eq. (1.1) with $a = E_{\min}$, $b = E_{\max}$, $K(x, y) = e^{-\sigma(y)x}$, where we have replaced E by y.

There are many difficulties inherent in solving Eq. (1.1). Let Eq. (1.1) be written in the form

$$g = Kf, \qquad Kf = \int_a^b K(x, y) f(y)\, dy.$$

We see that what is desired is an operator K^{-1} such that $f = K^{-1}g$. First, K may fail to have an inverse unless suitable restrictions are placed on $g(x)$. If K^{-1} does exist, it is an unbounded operator. This means that small errors in the data $g(x)$ may induce large errors in the solution $f(x)$.

It is this last point that we discuss. We examine two classical methods for solving (1.1), the collocation and Galerkin procedures, using both step-function and polynomial bases. In Section II, we show that both of these methods lead to matrix equations of the form

$$\hat{K}\hat{f} = \hat{g},$$

where the vector $\hat{f}$ is directly related to $f(x)$ and the vector $\hat{g}$ to $g(x)$. In Section III, we define the condition number of the matrix $\hat{K}$, which gives us a measure of the sensitivity of the solution $\hat{f}$ to errors in $\hat{g}$. (The larger the condition number, the greater the sensitivity.) In Section IV, we tabulate this condition number for various kernels and bases as a function of N, the number of basis functions used. The effect

that these ill-conditioned matrices have on the accuracy of solutions of (1.1) is demonstrated computationally in Section V. Our results are summarized in Section VI.

Motivation for this study has been provided in part by some new results on lower bounds for condition numbers of these matrices [1]. More specifically, it has been demonstrated there that the smoother the kernel $K(x, y)$ the more rapidly does the condition number grow as a function of the order of the matrix $\hat{K}$. Although the fact that smooth kernels lead to ill-conditioning of the corresponding matrices has long been observed, these observations have not previously been put in quantitative terms. We feel that the actual numerical calculation of the condition numbers for various problems provides valuable information concerning the usefulness of the theory developed in [1] and gives an indication of the efficacy of Galerkin and collocation methods in practical situations. Our results suggest that the theory may indeed help to predict the success or failure of these devices in any given problem. The computations reveal that for certain bases the condition number bounds are remarkably good.

II. The Galerkin and Collocation Procedures

In this section we describe the Galerkin and collocation methods for approximating the solution to the integral equation

$$g(x) = \int_a^b K(x, y) f(y)\, dy, \tag{2.1}$$

where a and b are finite and the kernel $K(x, y)$ is in L_2 on the square $a \leqslant x, y \leqslant b$. We assume the function $g(x)$ is given either analytically or as discrete data; in the latter case, it may be necessary to extend its definition.

In the Galerkin method we begin by selecting two orthonormal basis sets, $\{\vartheta_i\}$ and $\{\varphi_i\}$, and then approximating $f(y)$ and $g(x)$ as truncated series in these sets

$$f(y) \doteq \sum_{j=1}^{N} f_j \varphi_j(y), \tag{2.2a}$$

$$g(x) \doteq \sum_{i=1}^{N} g_i \vartheta_i(x). \tag{2.2b}$$

Substituting Eq. (2.2) into Eq. (2.1) and making use of the orthonormality of the ϑ_i gives

$$\begin{aligned} g_i &= \sum_{j=1}^{N} f_j \int_a^b \vartheta_i(x) \left\{ \int_a^b K(x, y)\, \varphi_j(y)\, dy \right\} dx \\ &= \sum_{j=1}^{N} (\vartheta_i, K\varphi_j) f_j, \qquad i = 1, 2, \ldots, N. \end{aligned} \tag{2.3}$$

Now define the vectors

$$\bar{f}=\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}, \qquad \bar{g}=\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix}$$

and the $N \times N$ matrix $\bar{K}$ by

$$\bar{K}=(K_{ij})=((\vartheta_i, K\varphi_j)). \tag{2.4}$$

Equation (2.3) becomes

$$\bar{K}\bar{f}=\bar{g}. \tag{2.5}$$

Because $g(x)$ is known, the components of $\bar{g}$ can be obtained from

$$g_i=(g, \vartheta_i)=\int_a^b g(x)\,\vartheta_i(x)\,dx, \qquad i=1,\ldots, N, \tag{2.6}$$

and the system of Eq. (2.5) solved for $\bar{f}$. The solution $f(x)$ can then be recovered from Eq. (2.2a).

It should be mentioned that the result described may be modified quite easily to accommodate the case of bases $\{\vartheta_i\}$ and $\{\varphi_i\}$ which are not orthonormal, a situation frequently arising in practice. The theory developed in [1], however, relies on the assumption of orthonormality in both Galerkin and collocation schemes; therefore we confine our investigation throughout this article to such cases.

Although the collocation method can be viewed as a Galerkin method (see [1]), we use the standard approach. Select an orthonormal set $\{\varphi_j\}$ and again expand $f(y)$ in the truncated series

$$f(y) \doteq \sum_{j=1}^{N} f_j \varphi_j(y). \tag{2.7}$$

Substitute this expression for $f(y)$ into Eq. (2.1) and evaluate the resulting equation at $x=x_i$, $i=1, 2,\ldots, N$, to obtain

$$g(x_i)=\sum_{j=1}^{N} f_j \int_a^b K(x_i, y)\,\varphi_j(y)\,dy, \qquad i=1, 2,\ldots, N.$$

Defining the vectors

$$\tilde{f}=\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}, \qquad \tilde{g}=\begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix},$$

and the matrix

$$\tilde{K} = (K_{ij}) = \left(\int_a^b K(x_i, y)\, \varphi_j(y)\, dy \right), \tag{2.8}$$

we obtain the equation

$$\tilde{K}\tilde{f} = \tilde{g}. \tag{2.9}$$

Solving Eq. (2.9) for $\tilde{f}$ we recover $f(x)$ from the expansion Eq. (2.7).

III. The Effect of Errors in the Data

Both of the methods discussed in Section II lead to the problem of solving linear algebraic systems of the form

$$\hat{K}\hat{f} = \hat{g}, \tag{3.1}$$

where $\hat{K}$ is an $n \times n$ matrix and $\hat{f}$, $\hat{g}$ are n-vectors. In this section we examine the condition number of the matrix $\hat{K}$ in an attempt to quantify how sensitive the solution $\hat{f}$ of Eq. (3.1) is to small errors in the data $\hat{g}$. In what follows, we assume that $\hat{K}$ is nonsingular.

Suppose $\hat{g}$ is subject to an error $\delta\hat{g}$, where $\delta\hat{g}$ is a vector of small norm. The corresponding vector $\hat{f} + \delta\hat{f}$ then satisfies

$$\hat{K}(\hat{f} + \delta\hat{f}) = \hat{g} + \delta\hat{g}. \tag{3.2}$$

It may be shown (see [2]) that

$$\frac{\|\delta\hat{f}\|}{\|\hat{f}\|} \leqslant \|\hat{K}\| \cdot \|\hat{K}^{-1}\| \cdot \frac{\|\delta\hat{g}\|}{\|\hat{g}\|}. \tag{3.3}$$

Here $\|\cdots\|$ denotes any acceptable norm. The *condition number* of $\hat{K}$ is defined by

$$\operatorname{cond}(\hat{K}) = \|\hat{K}\| \cdot \|\hat{K}^{-1}\|. \tag{3.4}$$

From Eq. (3.3) it is clear that $\operatorname{cond}(\hat{K})$ provides a bound on the relative error in $\hat{f}$ due to a given relative error in $\hat{g}$. Note that equality can actually hold in Eq. (3.3). The matrix $\hat{K} = \hat{I}$, the identity matrix, provides a trivial example. If $\hat{K}$ also contains errors, Eq. (3.3) can be modified. We do not discuss this case.

The value of $\operatorname{cond}(\hat{K})$ depends on the vector norm being used. If one uses the Euclidean vector norm, for example, the corresponding induced matrix norm is

$$\|\hat{K}\| = \sqrt{\rho(\hat{K}^T\hat{K})}.$$

where $\rho(\hat{K}^T\hat{K})$ is the spectral radius or largest eigenvalue of $\hat{K}^T\hat{K}$. In this norm

$$\text{cond}(\hat{K}) = \|\hat{K}\|\,\|\hat{K}^{-1}\| = \sigma_1/\sigma_n,$$

where

$$\sigma_1 = \sqrt{\rho(\hat{K}^T\hat{K})}, \qquad \sigma_n = 1/\sqrt{\rho((\hat{K}^{-1})^T(\hat{K}^{-1}))}.$$

(Note σ_n is also the smallest eigenvalue of $\hat{K}^T\hat{K}$.) It can be shown that if any norm other than the Euclidean norm is used, the value of $\text{cond}(\hat{K})$ is at least as great as σ_1/σ_n.

IV. Numerical Calculation of Some Condition Numbers

In this section we tabulate the condition numbers of the $\hat{K}$ matrix for both the Galerkin method, Eq. (2.4), and the collocation method, Eq. (2.8), for three different kernels $K(x, y)$, and two different sets of basis functions. Our study is not intended to be comprehensive but rather illustrative.

The kernels used are

(1) $K(x, y) = e^{-\alpha|x-y|}$, $-1 \leqslant x, y \leqslant 1$, $\alpha = 0.001, 0.1, 1.0, 10.0$,

(2) $K(x, y) = |x-y|^{\beta}$, $-1 \leqslant x, y \leqslant 1$, $\beta = 2.0, 2.001, 2.1, 2.5$, and

(3) $K(x, y) = (1-x)(1+y)$, $-1 \leqslant y \leqslant x \leqslant 1$,
$= (1-y)(1+x)$, $-1 \leqslant x \leqslant y \leqslant 1$.

Note that in cases 1 and 3, K is continuous but has a discontinuous first derivative. In case 2, K is smoother. The second derivative is continuous but not the third, except for the value $\beta = 2.0$ for which K possesses all derivatives. We expect these differences in smoothness to be reflected by the condition numbers.

The bases studied are (a) piecewise constant functions defined by

$$\varphi_i(x) = \sqrt{N/2}, \qquad -1 + \frac{2(i-1)}{N} \leqslant x \leqslant -1 + \frac{2i}{N}, \qquad i = 1, 2, \ldots, N$$

$$= 0, \qquad \text{elsewhere,}$$

and (b) Legendre polynomials.

Step functions are an especially convenient basis set for our numerical computations because the integrals in Eqs. (2.4), (2.6), and (2.8) need only be evaluated over subintervals of $[-1, 1]$ of length $2/N$. The Legendre polynomials are a different matter, however, and can be a source of numerical difficulty, particularly for large orders.

In the numerical results to follow, all integrals were evaluated using adaptive Newton–Cotes quadrature routines. All matrix condition numbers were estimated using the LINPACK routine SGECO [3]. Programs were written in Fortran and the

TABLE I

$K = \exp[-\alpha|x-y|]$ Galerkin Step Functions

N \ α	0.001	0.1	1.0	10.0
10	2.7(5)	2.6(3)	1.7(2)	3.9(0)
20	1.1(6)	1.1(4)	7.2(2)	1.3(0)
30	2.6(6)	2.5(4)	1.7(3)	2.7(1)
40	4.7(6)	4.5(4)	3.0(3)	4.8(1)
50	7.4(6)	7.0(4)	4.7(3)	7.5(1)
CN^{γ}	$2374N^{2.06}$	$22.6N^{2.06}$	$1.5N^{2.03}$	$0.05N^{1.83}$
L	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$

computations carried out on the CDC 7600 and Cray-1 computers at Los Alamos National Laboratory.

To gain more understanding about the behavior of the condition number as a function of N, the order of the matrix $\hat{K}$, we tabulated $\text{cond}(\hat{K})$ for a range of N values and then fitted the results to the function $\text{cond}(\hat{K}) = CN^{\gamma}$ in the sense of least squares. This functional form is suggested by the theory [1]. This fit is indicated in many of the tables. In some instances it is omitted because it seemed relatively meaningless, usually as a result of too few data points or very large condition

TABLE II

$K = \exp[-\alpha|x-y|]$ Collocation Step Functions

N \ α	0.001	0.1	1.0	10.0
20	7.7(5)	7.3(3)	4.9(2)	8.6(0)
40	3.1(6)	3.0(4)	2.0(3)	3.2(1)
60	7.1(6)	6.8(4)	4.5(3)	7.2(1)
80	1.3(7)	1.2(5)	8.0(3)	1.3(2)
100	2.0(7)	1.9(5)	1.3(4)	2.0(2)
CN^{γ}	$1738N^{2.03}$	$16N^{2.03}$	$1.1N^{2.03}$	$0.03N^{1.89}$
L	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$

TABLE III

$K = \exp[-\alpha|x-y|]$ Galerkin Legendre Polynomial

N \ α	0.001	0.1	1.0	10.0
2	2.5(3)	2.5(1)	2.5(0)	1.1(0)
4	2.1(4)	2.0(2)	1.5(0)	1.5(0)
6	7.2(4)	6.9(2)	4.8(1)	2.5(0)
8	1.8(5)	1.8(3)	1.2(2)	4.2(0)
10	4.0(5)	3.8(3)	2.6(2)	7.2(0)
CN^{γ}	Not meaningful			
L	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$

numbers. We also indicate by the letter L in each table the lower bound predicted by that theory:

$$L = cN^{\gamma},$$

where c is a generic constant usually dependent upon γ.

The different ranges of N values in Tables I–X were dictated by computational expense. In the tables the notation $p(q)$ means $p \times 10^{q}$. A single asterisk $(*)$ means that the calculated condition number was $O(10^{16})$, a magnitude we felt to be unreliable because of the limitations of machine arithmetic. A double asterisk $(**)$

TABLE IV

$K(x, y) = \exp\{-\alpha|x-y|\}$ Collocation Legendre Polynomials

N \ α	0.001	0.1	1.0	10.0
10	4.8(0)	4.5(4)	2.9(3)	1.1(2)
20	8.5(9)	8.0(7)	5.0(6)	1.2(5)
30	1.2(13)	1.1(11)	7.0(9)	1.4(8)
40	*	1.5(14)	9.0(12)	1.7(11)
50	*	*	*	2.1(14)
CN^{γ}	Not meaningful			
L	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$	$cN^{1.5}$

TABLE V

$K = |x - y|^{\beta}$ Galerkin Step Functions

N \ β	2.0	2.001	2.1	2.5
10	*	1.5(6)	1.8(4)	6.4(3)
20	*	1.3(7)	1.7(5)	1.1(5)
30	*	4.7(7)	6.5(5)	5.0(5)
40	*	1.2(8)	1.6(6)	1.4(6)
50	*	2.3(8)	3.4(6)	3.3(6)
CN^{γ}	—	$1058N^{3.15}$	$10.3N^{3.25}$	$0.9N^{3.86}$
L	cN^{γ}, γ arbitrary	$cN^{3.001}$	$cN^{3.1}$	$cN^{3.5}$

indicates a mathematically singular matrix. In the Galerkin method the same basis set was chosen for both the $f(y)$ and $g(x)$ expansions. In collocation the x_i's were equally spaced. In the tables, the principal entries are the condition numbers corresponding to the parameters indicated. The remaining entries are self-explanatory.

A careful examination of the tables shows that the smoother the kernel $K(x, y)$, the more ill-conditioned the problem, exactly as predicted by the theory. Another interesting point is that for the examples considered step function bases produce lower condition numbers than the polynomials.

TABLE VI

$K = |x - y|^{\beta}$ Collocation Step Functions

N \ β	2.0	2.001	2.1	2.5
20	*	8.8(6)	1.2(5)	7.7(4)
40	*	7.6(7)	1.1(6)	9.4(5)
60	*	2.6(8)	3.9(6)	4.1(6)
80	*	6.4(8)	9.7(6)	1.1(7)
100	*	1.3(9)	2.0(7)	2.5(7)
CN^{γ}	—	$806N^{3.10}$	$7.8N^{3.20}$	$1.02N^{3.71}$
L	cN^{γ}, γ arbitrary	$cN^{3.001}$	$cN^{3.1}$	$cN^{3.5}$

TABLE VII

$K = |x-y|^{\beta}$ Galerkin Legendre Polynomials

N \ β	2.0	2.001	2.1	2.5
2	1.0(0)	1.0(0)	1.0(0)	1.1(0)
4	**	2.5(4)	2.4(2)	4.9(1)
6	**	1.4(5)	1.6(3)	5.0(2)
8	**	5.3(5)	6.2(3)	2.3(3)
10	**	1.6(6)	1.9(4)	8.3(3)
CN^{γ}	Not meaningful			
L	cN^{γ}, γ arbitrary	$cN^{3.001}$	$cN^{3.1}$	$cN^{3.5}$

V. Numerical Results

To demonstrate that the phenomena discussed in Section III do actually occur in practice, we examined the integral equation

$$\frac{2}{\alpha}\left\{\left[x-\left(\frac{\alpha+1}{\alpha}\right)e^{-\alpha}\sinh(\alpha x)\right]+2[1-e^{-\alpha}\cosh(\alpha x)]\right\} = \int_{-1}^{1} e^{-\alpha|x-y|} f(y)\,dy \tag{5.1}$$

TABLE VIII

$K = |x-y|^{\beta}$ Collocation Legendre Polynomials

N \ β	2.0	2.001	2.1	2.5
10	*	8.4(6)	9.6(4)	3.5(4)
20	*	2.4(10)	2.8(8)	1.3(8)
30	*	4.3(13)	5.3(11)	2.6(11)
40	*	*	8.3(14)	4.5(14)
50	*	*	*	*
CN^{γ}	Not meaningful			
L	cN^{γ} γ arbitrary	$cN^{3.001}$	$cN^{3.1}$	$cN^{3.5}$

TABLE IX

$K = (1-x)(1+y), -1 \leqslant y \leqslant x \leqslant 1,$
$= (1-y)(1+x), -1 \leqslant x \leqslant y \leqslant 1,$ Step Functions

N	Galerkin	N	Collocation
10	1.5(2)	20	4.0(2)
20	6.0(2)	40	1.6(3)
30	1.3(3)	60	3.6(3)
40	2.4(3)	80	6.4(3)
50	3.7(3)	100	1.0(4)
CN^{γ}	$1.5N^{2.01}$		$1.0N^{2.0}$
L	$cN^{1.5}$		$cN^{1.5}$

for $\alpha = 0.001$ and $\alpha = 10.0$, using collocation and step functions. The vector $\hat{g}$ was first computed to machine accuracy (about 14 decimal digits) and $\hat{f}$ was then calculated. The result was compared to the analytic solution

$$f(y) = y + 2.0, \tag{5.2}$$

and the average relative error

$$\text{AVG} = \frac{1}{N} \sum_{j=1}^{N} \frac{|f_j - (y_j + 2)|}{|y_j + 2|} \tag{5.3}$$

was found.

TABLE X

$K = (1-x)(1+y), -1 \leqslant y \leqslant x \leqslant 1,$
$= (1-y)(1+x), -1 \leqslant x \leqslant y \leqslant 1,$ Legendre Polynomials

N	Galerkin	N	Collocation
2	4.3(0)	10	3.1(3)
4	3.8(1)	20	5.6(6)
6	1.3(2)	30	1.1(10)
8	3.5(2)	40	1.8(11)
10	7.6(2)	50	1.2(12)
CN^{γ}	Not meaningful		
L	$cN^{1.5}$		$cN^{1.5}$

TABLE XI

Values of AVG for $N = 100$

ε \ α	10.0	0.001
0	2.9(−5)	5.3(−5)
10^{-6}	7.4(−5)	5.6(0)
10^{-5}	5.0(−4)	5.6(1)
10^{-4}	4.9(−3)	5.6(2)

Next, error was introduced into the data function g by calculating

$$g_\varepsilon(x_l) = g(x_l)(1.0 + \varepsilon \text{ Rand}(i)), \tag{5.4}$$

where Rand(i) is a uniformly distributed random variable with value between −1.0 and 1.0. The process described was repeated for various values of ε.

Table XI presents the values of AVG obtained for $N = 100$. The fact that AVG is relatively large, even for $\varepsilon = 0.0$, is a bit surprising. An examination of the computed values of f_j indicates, however, that most of that error occurs in the vicinity of $x = \pm 1.0$. In the interior of the interval, relative errors of the order of 10^{-10} and less are found. This "end-point" effort becomes less and less significant as ε is increased.

It should be noted that the upper bound provided by Eq. (3.3) is remarkably good. For instance, with $\alpha = 10.0$ and $\varepsilon = 10^{-4}$, we find AVG $= 4.9 \times 10^{-3}$ and cond$(\hat{K}) = 2.0 \times 10^{2}$ (Table II). Thus the left side of Eq. (3.5) is 4.9×10^{-3} and the right is 2.0×10^{-2}. This same general behavior is exhibited by the other entries in Table XI provided $\varepsilon > 0$.

There is no assurance, of course, that such good agreement will be found in all problems. A wealth of interesting and valuable information has been revealed by extensive calculations of this kind, much of which cannot be presented here. It suggests strongly that the condition numbers are excellent indicators of "trends," and that they can be useful in predicting when a solution scheme will be reasonably successful and when it will fail completely.

Finally, it should be mentioned that often there is an optimal choice of N at which the most accurate solution is obtained when g contains error. This phenomenon has been observed in practice by many researchers, but seldom linked to matrix condition numbers.

VI. Summary and Conclusions

Two well-known methods for the numerical solution of integral equations of the first kind have been studied in a few reasonably representative cases. The condition

numbers of the related matrices have been estimated and compared with results provided by theory. Often these comparisons have been remarkably close.

We have shown by several examples that condition numbers provide a good indication of the reliability of the solution in the presence of noisy data. Thus, these condition numbers can help to predict the success or failure of the method.

There are, of course, many devices for the numerical solution of integral equations of the first kind (for example, see [4]). The technique of regularization is often used. In this article the order N serves as a regularization parameter. (For a more detailed discussion see [1]. In [5] a further regularization is carried out for the Galerkin method.) Each numerical algorithm has its advantages and its drawbacks. Often it is difficult to predict the success of a particular method. We believe the results of our investigation do provide, in some cases, such a priori information.

We note, too, that in many instances very satisfactory results have been obtained by the use of step-function bases and that the computing time and effort involved is usually much less than for the Legendre polynomials. Extensive experience, not reported here, indicates that unless the anticipated shape of the solution function f strongly suggests a particular choice of basis, the simple step functions are often the best choice.

REFERENCES

1. G. M. WING, Condition numbers of matrices arising from the numerical solution of linear equations of the first kind, *J. Integr. Equations*, to appear.
2. J. FRANKLIN, "Matrix Theory," Prentice–Hall, Englewood Cliffs, N.J., 1968.
3. J. J. DONGARRA, J. R. BUNCH, C. B. MOLER, AND G. W. STEWART, "LINPACK Users' Guide," SIAM, Philadelphia, Pa., 1979.
4. L. M. DELVES AND J. WALSH, "Numerical Solution of Integral Equations," Oxford Univ. Press (Clarendon), London/New York, 1974.
5. E. BABOLIAN AND L. M. DELVES, An augmented Galerkin matrix for first kind integral equations, *J. Inst. Math. Its Appl.* **24** (1979), 157–174.